

# On the Weil representation of infinite-dimensional symplectic group over a finite field

YU.A. NERETIN<sup>1</sup>

We extend the Weil representation of infinite-dimensional symplectic group to a representation a certain category of linear relations.

## 1 The statement of the paper

**1.1. The spaces  $\mathbb{V}_{2\mu}$ .** Denote by  $\mathbb{C}^\times$  be the multiplicative group of complex numbers. Denote by  $\mathbb{F}$  a finite field of characteristic  $p > 2$ , by  $\mathbb{F}^\times$  its multiplicative group. Fix a nontrivial character  $\text{Exp}(\cdot)$  of the additive group of  $\mathbb{F}$ ,

$$\text{Exp}(a + b) = \text{Exp}(a) \text{Exp}(b), \quad \text{Exp}(\cdot) \in \mathbb{C}^\times.$$

Values of  $\text{Exp}(\cdot)$  are  $e^{2\pi i k/p}$ .

For  $n = 0, 1, \dots$ , we denote by  $V_n^d$  an  $n$ -dimensional linear space over  $\mathbb{F}$ , to be definite we assume that  $V_n^d$  is the coordinate space  $\mathbb{F}^n$ . Denote by  $V_n^c$  the second copy of this space.

Next, we define the space  $V_\infty^d$  as a direct sum of a countable number of  $\mathbb{F}$  and the space  $V_\infty^c$  as a direct product of a countable number of  $\mathbb{F}$ . We equip the first space by the discrete topology, the second space by the product-topology (' $d$ ' is an abbreviation of 'discrete' and ' $c$ ' of 'compact').

Below subscripts  $\mu, \nu, \varkappa$  denote  $0, 1, \dots, \infty$ .

A space  $V_\mu^c$  is the Pontryagin dual of  $V_\mu^d$ , the pairing is given by

$$[x, y] := \sum_{j=1}^{\mu} x_j y_j.$$

(actually, the sum is finite). For a subspace  $L \subset V_\mu^d$  we denote by  $L^\circ$  the *annihilator* of  $L$  in  $V_\mu^c$  and vice versa.

Denote by  $\mathbb{V}_{2\mu}$  the space

$$\mathbb{V}_{2\mu} := V_\mu^d \oplus V_\mu^c$$

equipped with the following skew-symmetric bilinear form (*symplectic form*):

$$\{(x, y), (x', y')\} = \sum_{j=1}^{\mu} (x_j y'_j - y_j x'_j).$$

---

<sup>1</sup>Supported by the grant FWF, Project P28421.

The space  $\mathbb{V}_{2\infty}$  is a locally compact Abelian group and we can apply the Pontryagin duality.

For a subspace  $R \subset \mathbb{V}_{2\mu}$  we denote by  $R^\diamond$  the *orthocomplement* with respect to  $\{\cdot, \cdot\}$ . By the Pontryagin duality (see [7], Proposition 38), for closed subspaces  $X, Y \subset \mathbb{V}_{2\infty}$  the conditions  $X = Y^\diamond$  and  $Y = X^\diamond$  are equivalent (in other words for a closed subspace  $X$  we have  $X^{\diamond\diamond} = X$ ).

Denote by  $\text{Sp}(\mathbb{V}_{2\mu})$  the group of invertible continuous operators in  $\mathbb{V}_{2\mu}$  preserving the form  $\{\cdot, \cdot\}$ .

**1.2. The Weil representation of  $\text{Sp}(\mathbb{V}_{2\infty})$ .** Consider the space  $\ell^2(V_\mu^d)$  on the set  $V_\mu^d$ . For any  $v = (v^d, v^c) \in \mathbb{V}_{2\mu}$  we define a unitary operator  $a(v)$  in  $\ell^2(V_\mu^d)$  given by

$$a(v)f(x) = f(x + v^d) \text{Exp} \left[ \sum (x_j v_j^c + \frac{1}{2} v_j^d v_j^c) \right].$$

We have

$$a(v)a(w) = \text{Exp}(\frac{1}{2}\{v, w\}) a(v + w).$$

Thus we get a projective unitary representation of the additive group  $\mathbb{V}_{2\mu}$  or a linear representation of the *Heisenberg group*, the latter group is  $\mathbb{V}_{2\mu} \oplus \mathbb{F}$  with a product

$$(v; s) \star (w; t) = (v + w; s + t + \frac{1}{2}\{v, w\}).$$

**Proposition 1.1** *For any  $g \in \text{Sp}(\mathbb{V}_{2\infty})$  there exists a unique up to a scalar factor unitary operator  $W(g)$  in  $\ell^2(V_\infty^d)$  such that*

$$a(vg) = W(g)^{-1}a(v)W(g) \tag{1.1}$$

and

$$W(g_1)W(g_2) = c(g_1, g_2)W(g_1g_2), \quad \text{where } c(g_1, g_2) \in \mathbb{C}, |c(g_1, g_2)| = 1.$$

This statement is inside the original construction of A. Weil (the group  $V_\infty^d$  is locally compact), in Section 2 we present a proof (this proof is used in Section 3).

REMARK. The author does not know is the Weil representation of  $\text{Sp}(\mathbb{V}_{2\infty})$  projective or linear, for a finite symplectic group  $\text{Sp}(2n, \mathbb{F})$  it is linear, see, e.g., [4].  $\square$

**1.3. Linear relations.** Let  $X, Y$  be linear spaces over  $\mathbb{F}$ . A *linear relation*  $T : X \rightrightarrows Y$  is a linear subspace in  $X \oplus Y$ . Let  $T : X \rightrightarrows Y$ ,  $S : Y \rightrightarrows Z$  be linear relations. Their product  $ST$  is defined as the set of all  $(x, z) \in X \oplus Z$  such that there exists  $y \in Y$  satisfying  $(x, y) \in T$ ,  $(y, z) \in S$ .

REMARK. Let  $A : X \rightarrow Y$  be a linear operator. Then its graph  $\text{graph}(A)$  is a linear relation  $X \rightrightarrows Y$ . If  $A : X \rightarrow Y$ ,  $Y \rightarrow Z$  are linear operators, then product of their graphs

$$\text{graph}(A) : X \rightrightarrows Y, \quad \text{graph}(B) : Y \rightrightarrows Z$$

is the linear relation  $\text{graph}(BA) : X \rightrightarrows Z$ .  $\square$

For a linear relation  $T : X \rightrightarrows Y$  we define:

- the *kernel*  $\ker T$  is the intersection  $T \cap (X \oplus 0)$ ;
- the *domain*  $\text{dom } T$  is the image of the projection of  $T$  to  $X \oplus 0$ ;
- the *image*  $\text{im } T$  is the image of the projection of  $T$  to  $0 \oplus Y$ ;
- the *indefiniteness*  $\text{indef } T$  is the intersection  $T \cap (0 \oplus Y)$ .

We define the *pseudo-inverse relation*  $T^\square : Y \rightrightarrows X$  as the image of  $T$  under the transposition of summands in  $X \oplus Y$ .

For a linear relation  $T : X \rightrightarrows Y$  and a subspace  $U \subset X$  we define a subspace  $TU \subset Y$  consisting of all  $y \in Y$  such that there exists  $u \in U$  satisfying  $(u, y) \in T$ .

REMARK. For a linear operator  $A : X \rightarrow Y$  we have

$$\begin{aligned} \ker \text{graph}(A) &= \ker A, & \text{dom } \text{graph}(A) &= X, \\ \text{im } \text{graph}(A) &= \text{im}(A), & \text{indef } \text{graph}(A) &= 0. \end{aligned}$$

If  $A$  is invertible, then  $\text{graph}(A)^\square = \text{graph}(A^{-1})$ . For a subspace  $U \subset X$  we have  $\text{graph}(A)U = AU$ .  $\square$

**1.4. Perfect Lagrangian linear relations.** We say that a subspace  $Z \subset \mathbb{V}_{2\infty}$  is *codiscrete* if it is closed and the quotient  $\mathbb{V}_{2\infty}/Z$  is discrete.

Let  $\mu, \nu = 0, 1, \dots, \infty$ . Define a skew-symmetric bilinear form in  $\mathbb{V}_{2\mu} \oplus \mathbb{V}_{2\nu}$  by

$$\{(v, w), (v', w')\}_{\mathbb{V}_{2\mu} \oplus \mathbb{V}_{2\nu}} = \{v, v'\}_{\mathbb{V}_{2\mu}} - \{w, w'\}_{\mathbb{V}_{2\nu}}.$$

We say that a linear relation  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$  is *perfect Lagrangian* if it satisfies the following conditions:

- 1) subspace  $T$  is maximal isotropic in  $\mathbb{V}_{2\mu} \oplus \mathbb{V}_{2\nu}$  (in particular,  $T$  is closed);
- 2)  $\ker T$  and  $\text{indef } T$  are compact;
- 3)  $\text{im } T, \text{dom } T$  are codiscrete.

REMARK. Let  $A \in \text{Sp}(\mathbb{V}_{2\mu})$ . Then  $\text{graph}(A)$  is a perfect Lagrangian linear relation  $\mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\mu}$ .  $\square$

**Lemma 1.2** For any perfect Lagrangian linear relation  $T$ ,

$$\begin{aligned} (\text{dom } T)^\diamond &= \ker T, & (\ker T)^\diamond &= \text{dom } T, \\ (\text{im } T)^\diamond &= \text{indef } T, & (\text{indef } T)^\diamond &= \text{im } T. \end{aligned}$$

**Lemma 1.3** Let  $T : \mathbb{V}_{2\mu} \rightarrow \mathbb{V}_{2\nu}$  be a perfect Lagrangian linear relation and  $\ker T = 0, \text{indef } T = 0$ . Then  $\mu = \nu$  and  $T$  is a graph of an element of  $\text{Sp}(\mathbb{V}_{2\mu})$ .

**Theorem 1.4** If linear relations  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}, S : \mathbb{V}_{2\nu} \rightrightarrows \mathbb{V}_{2\kappa}$  are perfect Lagrangian, then  $ST : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\kappa}$  is perfect Lagrangian.

We define a category  $\mathcal{L}$ , whose objects are the spaces  $\mathbb{V}_{2\mu}$  and morphisms are perfect Lagrangian relations. It is more convenient to think that objects of the category are linear spaces, which are equivalent to the spaces  $\mathbb{V}_{2\mu}$  as linear spaces equipped with topology and skew-symmetric bilinear form.

### 1.5. Extension of the Weil representation.

**Theorem 1.5** a) *Let  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$  be a perfect Lagrangian linear relation. Then there exists a unique up to a scalar factor non-zero bounded operator  $W(T) : \ell^2(V_\mu^d) \rightarrow \ell^2(V_\nu^d)$  such that*

$$a(w)W(T) = W(T)a(v), \quad \text{for any } (v, w) \in T. \quad (1.2)$$

b) *For any perfect Lagrangian linear relations  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$ ,  $T : \mathbb{V}_{2\nu} \rightrightarrows \mathbb{V}_{2\kappa}$ ,*

$$W(S)W(T) = c(T, S)W(ST), \quad \text{where } c(T, S) \in \mathbb{C}^\times.$$

c) *For any perfect Lagrangian  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$ ,*

$$W(T)^* = W(T^\square).$$

**1.6. Gaussian operators.** Let  $H \subset V_\mu^d \oplus V_\nu^d$  be a subspace such that projection operators  $H \rightarrow V_\nu^d \oplus 0$  and  $H \rightarrow 0 \oplus V_\mu^d$  have finite dimensional kernels. Let  $Q$  be a quadratic form on  $H$ . We define a Gaussian operator  $G(H; Q) : \ell^2(V_\mu^d) \rightarrow \ell^2(V_\nu^d)$  by the formula

$$G(H; Q)f(x) = \sum_{y: (y, x) \in H} \text{Exp}(Q(x, y)) f(y). \quad (1.3)$$

**Theorem 1.6** *Any operator  $W(T) : \ell^2(V_\mu^d) \rightarrow \ell^2(V_\nu^d)$  is a Gaussian operator up to a scalar factor. Any Gaussian operator has such a form.*

**1.7. Bibliographic remarks.** 1) In 1951, K. O. Friedrichs [3] noticed that the Stone-von Neumann theorem implies the existence of a representation of a real symplectic group  $\text{Sp}(2n, \mathbb{R})$ . The finite-dimensional groups  $\text{Sp}(2n, \mathbb{R})$  were not interesting to him and he formulated a conjecture about an infinite-dimensional symplectic group. The representation of  $\text{Sp}(2n, \mathbb{R})$  was explicitly described by I. Segal in 1959, [16]. A. Weil extended this construction to groups over  $p$ -adic and finite fields.

2) For infinite-dimensional symplectic group the 'Weil representation' (which was conjectured by Friedrichs) was constructed by D. Shale [17] and F. A. Berezin [1], [2]. The  $p$ -adic case was examined by M. L. Nazarov [9], [8] and E. I. Zelenov [19].

3) The third part of the topic was a categorization of the Weil representation, it had a long pre-history (see [6]), for real and  $p$ -adic infinite-dimensional groups it was obtained by M.L.Nazarov, G.I.Olshanski, and the author in [9], 1989 (see more detailed expositions of different topics in [10], [8], [14], [11]).

For  $\mathrm{Sp}(2n, \mathbb{F})$  the corresponding representation of the semigroup of linear relations was done in the preprint of R.Howe (1976), which was not published as a paper. A detailed exposition of this representation is contained in<sup>2</sup> [12], Section 9.4.

4) Now there exists well-developed representation theory of infinite-dimensional classical groups and infinite symmetric groups. Numerous efforts to develop a parallel representation theory of infinite-dimensional classical groups over finite fields met some obstacles, which apparently, were firstly observed by Olshanski in [14], 1991. For a collection of references on different approaches to this topic, see [13]). In [13] there was proposed a version of  $\mathrm{GL}(\infty)$  over finite fields as a group of continuous operators in  $\mathbb{V}_{2\infty}$ , it seems that this group is an interesting and hand object from the point of view of representation theory. In particular, there is a natural  $\mathrm{GL}(\mathbb{V}_{2\infty})$ -invariant measure on an infinite-dimensional Grassmannian (and also on flag spaces) over  $\mathbb{F}$  and explicit decomposition of  $L^2$  on this Grassmannian. Clearly, this topic is related to the group  $\mathrm{Sp}(\mathbb{V}_{2\infty})$  and this requires an examination of the Weil representation.

**1.8. Acknowledges.** This note is a kind of addition to the work by Nazarov, Neretin, and Olshanski [9], [10], [8], [14]. I am grateful to M.L.Nazarov and G.I.Olshanski. I also thank R.Howe who recently sent me preprint [5].

**1.9. Structure of the paper.** In Section 2 we prove Proposition 1.1. The proof follows I.Segal arguments [16], see also [12], Section 1.2. The only additional difficulty is Proposition 2.2 about generators of the group  $\mathrm{Sp}(\mathbb{V}_{2\infty})$ .

Statements on linear relations are established in Section 3. A proof of the theorem about product of perfect Lagrangian relations is based on all the remaining statements, this makes a structure of the section slightly sophisticated. Lemmas 1.2 and 1.3 are proved in Subs. 3.1–3.3, Theorem 1.5.a in Subs. 3.4. Properties of Gaussian operators (Theorem 1.6) are established in Subs. 3.6–3.8. Theorem 1.4 is proved in Subs. 3.9. The remaining part of Theorem 1.5 is proved in Subs. 3.10.

## 2 The Weil representation of the group $\mathrm{Sp}(\mathbb{V}_{2\infty})$ .

Here we prove Proposition 1.1.

**2.1. Preliminary remarks on linear transformations in  $\mathbb{V}_{2\infty}$ .** Denote by  $\mathrm{GL}(\mathbb{V}_{2\infty})$  the group of all continuous invertible linear transformations  $v \mapsto vg$  of  $\mathbb{V}_{2\infty}$ . Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\mathbb{V}_{2\infty}). \quad (2.1)$$

Then the block  $c$  contains only finite number of nonzero elements; each row of  $a$  contains only a finite number of nonzero elements, and each column of  $d$  contains a finite number of nonzero elements.

We need some statements from [10], Sect. 2.

---

<sup>2</sup>I referred to [9] being sure that case of finite fields was considered in that paper.

**A.** Let  $g : \mathbb{V}_{2\infty} \rightarrow \mathbb{V}_{2\infty}$  be a continuous surjective linear transformation in  $\mathbb{V}_{2\infty}$ . Then the inverse transformation is continuous.

**B.** We say that a continuous linear transformation  $P$  in  $V_\infty^d$  or  $V_\infty^c$  is *Fredholm* if  $\dim \ker P < \infty$ ,  $\text{im } P$  is closed (this condition is nontrivial only for  $V_\infty^d$ ), and  $\text{codim im } P < \infty$ . The *index* of a Fredholm operator  $P$  is

$$\text{ind } P := \dim \ker P - \text{codim im } P.$$

The following statements hold:

1. An operator  $P$  in  $V_\infty^c$  is Fredholm iff the dual operator  $P^t$  in  $V_\infty^d$  is Fredholm.

2. Let  $P$  be Fredholm and  $H$  have a finite rank. Then  $P + H$  is Fredholm and

$$\text{ind}(P + H) = \text{ind } P.$$

3. If  $P, Q$  are Fredholm linear transformations of  $V_\infty^c$  or  $V_\infty^d$ , then

$$\text{ind } PQ = \text{ind } P + \text{ind } Q.$$

**C.** For any matrix  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(\mathbb{V}_{2\infty})$  (see (2.1)) the blocks  $a, d$  are Fredholm and  $\text{ind } d = -\text{ind } a$ .

**2.2. Fredholm indices of blocks of  $g \in \text{Sp}(\mathbb{V}_{2\mu})$ .** Elements  $g \in \text{Sp}(\mathbb{V}_{2\mu}) \subset \text{GL}(\mathbb{V}_{2\mu})$  satisfy obvious additional conditions, below we refer to

$$c^t a = a c^t; \tag{2.2}$$

$$a d^t - b c^t = 1. \tag{2.3}$$

**Lemma 2.1** Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(\mathbb{V}_{2\mu})$ . Then  $\text{ind } a = \text{ind } d = 0$ .

PROOF. For  $g \in \text{Sp}(\mathbb{V}_{2\mu})$ , we have

$$\text{ind}(a d^t) = \text{ind}(a) + \text{ind}(d^t) = \text{ind}(a) - \text{ind}(d) = 2 \text{ind}(a).$$

On the other hand,  $b c^t$  has finite rank. By (2.3),

$$\text{ind}(a d^t) = \text{ind}(1 + b c^t) = \text{ind}(1) = 0.$$

Thus,  $\text{ind}(a) = 0$ . □

**2.3. Generators of the group  $\text{Sp}(\mathbb{V}_{2\infty})$ .** We define the following subgroups in  $\text{Sp}(\mathbb{V}_{2\infty})$ :

- the subgroup  $H$  consists of matrices of the form  $\begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix}$ ;
- the subgroup  $N_+$  consists of matrices of the form  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ , where  $b = b^t$ ;

— the subgroup  $N_-$  consists of matrices of the form  $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ , where  $c = c^t$ .

Denote by  $J_k$  the following block matrix of the size  $((k-1)+1+\infty) + ((k-1)+1+\infty)$ :

$$J_k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Proposition 2.2** *The group  $\mathrm{Sp}(\mathbb{V}_{2\infty})$  is generated by the subgroup  $H$ ,  $N_+$ , and elements  $J_k$ .*

**2.4. Generators of  $\mathrm{Sp}(\mathbb{V}_{2\infty})$ . Proof of Proposition 2.2.** Let  $s = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in N_-$ . Since  $Q$  contains only a finite number of nonzero elements, we have

$$J_1 \dots J_m s J_m^{-1} \dots J_1^{-1} \in N_+$$

for sufficiently large  $m$ . Thus,  $N_-$  is contained in the subgroup generated by  $H$ ,  $N_+$ , and  $J_k$ .

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(\mathbb{V}_{2\infty}).$$

We will multiply it by elements of groups  $H$ ,  $N_+$ ,  $N_-$  and elements  $J_k$  in an appropriate way. As a result we will come to the unit matrix.

Recall that  $a$  is Fredholm of index 0. Therefore there are invertible operators  $K, L$  in  $V_\infty^d$  such that  $KaL$  is a block  $(l+\infty) \times (l+\infty)$ -matrix of the form  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . This follows from [13], Lemma 2.7. We pass to a new matrix  $g'$  given by

$$g' := \begin{pmatrix} K & 0 \\ 0 & K^{t-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{t-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & b'_{11} & b'_{12} \\ 0 & 1 & b'_{21} & b'_{22} \\ c'_{11} & c'_{12} & d'_{11} & d'_{12} \\ c'_{21} & c'_{22} & d'_{21} & d'_{22} \end{pmatrix}$$

(the size of the right hand side is  $l+\infty+l+\infty$ ). The condition (2.2) implies  $c'_{12} = 0$ ,  $c'_{21} = 0$ ,  $(c'_{22})^t = c'_{22}$ . In particular the following matrix

$$s := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c'_{22} & 1 \end{pmatrix}$$

is contained in  $N_-$ . The element  $sg'$  has the form

$$sg' = \begin{pmatrix} 0 & 0 & b''_{11} & b''_{12} \\ 0 & 1 & b''_{21} & b''_{22} \\ c''_{11} & 0 & d''_{11} & d''_{12} \\ 0 & 0 & d''_{21} & d''_{22} \end{pmatrix} =: g''.$$

The matrix  $c''_{11}$  is nondegenerate (otherwise, the whole matrix  $sg'$  is degenerate). We take an element

$$J_1 \dots J_l g'' =: g'''$$

and get a matrix of the form

$$g''' = \begin{pmatrix} -c'''_{11} & 0 & b'''_{11} & b'''_{12} \\ 0 & 1 & b'''_{21} & b'''_{22} \\ 0 & 0 & d'''_{11} & d'''_{12} \\ 0 & 0 & d'''_{21} & d'''_{22} \end{pmatrix}.$$

Keeping in the mind (2.3) we observe

$$\begin{pmatrix} d'''_{11} & d'''_{12} \\ d'''_{21} & d'''_{22} \end{pmatrix} = \begin{pmatrix} -c'''_{11} & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Element

$$\begin{pmatrix} -c'''_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(c'''_{11})^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} g'''$$

is contained in  $N_+$ . □

**2.5. Construction of the Weil representation of the symplectic groups  $\mathrm{Sp}(\mathbb{V}_{2\infty})$ .** A remaining part of a proof of Proposition 1.1 is based on standard arguments (see [12], Sect. 1.2), which were proposed by I. Segal).

STEP 1. *The representation  $a(v)$  of the Heisenberg group in  $\ell^2(V_\infty^d)$  is irreducible.* Indeed, let us show that there are no nontrivial intertwining operators. Any bounded operator  $Q$  in  $\ell^2(V_\infty^d)$  commuting with all operators

$$a(0, v^c)f(x) = \mathrm{Exp}\left(\sum v_j^c x_j\right) f(x)$$

is an operator of multiplication by a function. Since  $Q$  commutes also with shifts

$$a(v^d, 0)f(x) = f(x + v^d),$$

we get that  $Q$  is a multiplication by a constant.

STEP 2. *An operator  $W(g)$  is defined up to a scalar factor (if it exists).* Indeed, the map  $v \mapsto vg$  is an automorphism of the Heisenberg group. Therefore, the formula  $v \mapsto a(vg)$  determines a unitary representation of the Heisenberg



group. The operator  $W(g)$  intertwines unitary representations  $a(v)$  and  $a(vg)$ . By the Schur lemma,  $W(g)$  is unique up to a scalar factor.

STEP 3. If  $W(g_1)$ ,  $W(g_2)$  exist, then

$$W(g_1)W(g_2) = \lambda \cdot W(g_1g_2).$$

Indeed,

$$\begin{aligned} (W(g_1)W(g_2))^{-1}a(v)W(g_1)W(g_2) &= W(g_2)^{-1}(W(g_1)^{-1}a(v)W(g_1))W(g_2) = \\ &= W(g_2)^{-1}a(vg_1)W(g_2) = a(vg_1g_2). \end{aligned}$$

STEP 4. It remains to write operators corresponding to generators of the group  $\mathrm{Sp}(\mathbb{V}_{2\mu})$ .

— Operators corresponding to elements of the subgroup  $H$  are

$$W \begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix} f(x) = f(xa);$$

— Operators corresponding to elements of  $N_+$  are

$$W \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(x) = \mathrm{Exp}(\frac{1}{2} \sum b_{kl} x_k x_l) f(x). \quad (2.4)$$

— An operator corresponding to  $J_k$  is the Fourier transform with respect to the coordinate  $x_k$ .

### 3 The Weil representation of the category of Lagrangian relations

#### 3.1. On a canonical form of a compact isotropic submodule.

**Lemma 3.1** *For any compact isotropic subspace  $M \subset \mathbb{V}_{2\infty}$  there is an element  $g \in \mathrm{Sp}(\mathbb{V}_{2\infty})$  such that  $Mg \subset V_\infty^c$ . Moreover, we can choose  $g$  in such a way that  $Mg \subset V_\infty^c$  be a subspace given by a system of equations of the type  $y_\alpha = 0$ , where  $\alpha$  ranges in a subset  $A \subset \mathbb{N}$ .*

PROOF. Consider the projection map  $\pi : M \rightarrow V_\infty^c$ . Its fibers are compact and discrete, therefore they are finite. In particular,  $\pi^{-1}(0)$  is a finite-dimensional subspace in  $V_\infty^d$ . We can choose an element  $h$  of the subgroup  $H \subset \mathrm{Sp}(\mathbb{V}_{2\infty})$  such that  $(\pi^{-1}(0))h \subset V_\infty^d$  is the subspace consisting of vectors  $(x_1, \dots, x_k, 0, \dots)$ . Since  $Mh$  is isotropic,  $\pi(Mh)$  is contained in the subspace  $y_1 = \dots = y_k = 0$ . Therefore  $MhJ_1 \dots J_k \subset V_\infty^c$ .

Next, we wish to describe closed subspaces in  $V_\infty^c$  modulo linear transformations of  $V_\infty^c$  (they are induced by elements of the subgroup  $H$ ). The Pontryagin duality determines a one-to-one correspondence between sets of closed subgroups in Abelian groups  $V_\infty^c$  and  $V_\infty^d$  (see [7], Theorem 27); the both groups

are equipped with an Abelian group of automorphisms  $x \mapsto \lambda x$ , where  $\lambda \in \mathbb{F}^\times$ . It is easy to see that the correspondence send invariant subgroups (subspaces) to invariant subgroups. Therefore the question is reduced to a description of subspaces in  $V_\infty^d$  modulo linear transformations, the latter problem is trivial.  $\square$

**Corollary 3.2** *Let  $L$  be a compact isotropic submodule of  $\mathbb{V}_{2\infty}$ . Then the space  $L^\diamond/L$  is isomorphic as a symplectic space to  $\mathbb{V}_{2n}$  or  $\mathbb{V}_{2\infty}$ .*

**Lemma 3.3** *For any codiscrete coisotropic submodule  $L \subset \mathbb{V}_{2\infty}$  there exists  $g \in \text{Sp}(\mathbb{V}_{2\infty})$  such that  $Lg \supset V_\infty^c$ .*

PROOF. We reduce the compact isotropic module  $L^\diamond$  to the canonical form (i.e.,  $L^\diamond$  is a coordinate subspace in  $V_\infty^c$ ).  $\square$

**3.2. Proof of Lemma 1.2.** Consider a perfect Lagrangian relation  $T$ . Obviously, for  $v \in \ker T$ ,  $Z \in \text{dom } T$ , we have  $\{v, z\} = 0$ , i.e.,  $(\text{dom } T)^\diamond \supset \ker T$ . Next,  $(\text{dom } T)^\diamond = \ker T$ ; otherwise, we take an isotropic linear relation  $T + (\text{dom } T)^\diamond \supsetneq T$ .  $\square$

**3.3. Proof of Lemma 1.3.** Only the case  $\mu = \nu = \infty$  requires a proof. Let  $T : \mathbb{V}_{2\infty} \rightrightarrows \mathbb{V}_{2\infty}$  be a perfect Lagrangian linear relation,  $\ker T = 0$ ,  $\text{indef } T = 0$ . According Lemma 1.2, we have  $\text{dom } T = \mathbb{V}_{2\infty}$ ,  $\text{im } T = \mathbb{V}_{2\infty}$ .

**Lemma 3.4** *The subspace  $T$  is isomorphic to  $\mathbb{V}_{2\infty}$  as a linear space with a topology.*

PROOF OF LEMMA 1.3. We apply statement **A** of Subsect. 2.1. The projections

$$\pi_T : T \rightarrow \mathbb{V}_{2\infty} \oplus 0, \quad \pi'_T : T \rightarrow 0 \oplus \mathbb{V}_{2\infty}$$

are continuous, therefore the inverse map  $\pi_T^{-1}$  is continuous. Hence the map  $\pi'_T \circ \pi_T^{-1}$  is continuous, this is the the linear transformation whose graph is  $T$ .  $\square$

PROOF OF LEMMA 3.4. Represent  $\mathbb{V}_{2\infty}$  as a union of a chain of compact subspaces

$$W_0 = V_\infty^c \subset W_1 \subset W_2 \subset \dots$$

Consider the projection map  $\pi : \mathbb{V}_{2\infty} \oplus \mathbb{V}_{2\infty} \rightarrow \mathbb{V}_{2\infty} \oplus 0$ , denote by  $\pi_T$  its restriction to  $T$ . Then

$$\pi_T^{-1} V_\infty^c = \cup_{j=1}^\infty [T \cap (V_\infty^c \oplus W_j)].$$

Therefore

$$V_\infty^c = \cup_{j=1}^\infty \pi_T [T \cap (V_\infty^c \oplus W_j)].$$

In the right hand side we have a union of an increasing sequence of compact sets, the left hand side is compact. Therefore for some  $k$ ,

$$V_\infty^c = \pi_T [T \cap (V_\infty^c \oplus W_k)].$$

The set  $T \cap (V_\infty^c \oplus W_k)$  is compact and  $\pi_T$  is continuous, therefore the inverse map  $\pi_T^{-1} : V_\infty^c \rightarrow T \cap (V_\infty^c \oplus W_k)$  is continuous.

Next, denote by  $e_l$  the standard basis in  $V_\infty^d$ . Then

$$T \simeq [T \cap (V_\infty^c \oplus W_k)] \oplus \bigoplus_{l=1}^{\infty} \mathbb{F} \cdot \pi_T^{-1} e_l.$$

Thus  $T$  is isomorphic  $\mathbb{V}_{2\infty} \oplus 0$ . □

### 3.4. Proof of Theorem 1.5.a. Existence of operators $W(T)$ .

**Lemma 3.5** *Let the claim of Theorem 1.5.a hold for a linear relation  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$ . Then the same statement holds for any linear relation  $gTh$ , where  $g \in \text{Sp}(\mathbb{V}_{2\nu})$ ,  $h \in \text{Sp}(\mathbb{V}_{2\mu})$ .*

PROOF. Obvious. □

We start a proof of the theorem. Keeping in the mind Lemma 3.1 we can assume

$$\ker T \subset V_\mu^c, \quad \text{indef } T \subset V_\nu^c.$$

Let  $w \in \text{indef } T$ . Then

$$a(w)W(T) = W(T)a(0) = W(T).$$

The operator  $a(w)$  is an operator of multiplication by a function,

$$a(w)f(x) = \text{Exp}\left(\sum x_j w_j^c\right)f(x)$$

Therefore any function  $\psi \in \text{im } W(T)$  vanishes on the set  $\text{Exp}(\sum x_j w_j^c) \neq 1$ . In other words, all elements of  $\psi \in \text{im } W(T)$  are supported by the subspace  $(\text{indef } T)^\circ \subset V_\nu^d$ .

Let  $v \in \ker T$ . Then

$$W(T) = W(T)a(v).$$

The operator  $a(v)$  is a multiplication by function taking finite number of values  $\lambda_l = \exp(2\pi li/p)$ . Therefore,  $\lambda_l$  are the eigenvalues of  $a(v)$ , and  $T(W) = 0$  on all subspaces  $\ker(a(v) - \lambda_l)$  for  $\lambda_l \neq 1$ . Therefore,  $W(T)$  is zero for any function supported by  $V_\mu^d \setminus (\ker T)^\circ$ .

Consider decompositions

$$\begin{aligned} \ell_2(V_\mu^d) &= \ell_2((\ker T)^\circ) \oplus \ell_2(V_\mu^d \setminus (\ker T)^\circ); \\ \ell_2(V_\nu^d) &= \ell_2((\text{indef } T)^\circ) \oplus \ell_2(V_\nu^d \setminus (\text{indef } T)^\circ). \end{aligned}$$

The operator  $W(T)$  has the following block form with respect to this decomposition

$$W(T) = \begin{pmatrix} \widetilde{W}(T) & 0 \\ 0 & 0 \end{pmatrix},$$

with a non-zero block

$$\widetilde{W}(T) : \ell^2((\ker T)^\circ) \rightarrow \ell^2((\operatorname{indef} T)^\circ).$$

The linear relation  $T$  determines a linear relation  $T' : \operatorname{dom} T \rightrightarrows \operatorname{im} T$ . We take the projection

$$\operatorname{dom} T \oplus \operatorname{im} T \rightarrow (\operatorname{dom} T / \ker T) \oplus (\operatorname{im} T / \operatorname{indef} T)$$

and the image  $\widetilde{T}$  of  $T'$  under this projection. Thus we get a linear relation  $\widetilde{T} : \operatorname{dom} T / \ker T \rightrightarrows \operatorname{im} T / \operatorname{indef} T$ . The spaces  $\operatorname{dom} T / \ker T$ ,  $\operatorname{im} T / \operatorname{indef} T$  have form  $\mathbb{V}_{2\kappa}$ . By construction,  $\ker \widetilde{T} = 0$ ,  $\operatorname{indef} \widetilde{T} = 0$ . Therefore,  $\widetilde{T}$  is a graph of a symplectic operator in  $\mathbb{V}_{2\kappa}$ .

It is easy to see that  $\widetilde{W}(T)$  has the same commutation relations with the Heisenberg group as  $W(\widetilde{T})$ . It remains to refer to Proposition 1.1.  $\square$

**3.5. One corollary from the previous Subsection.** For a linear embedding  $B : V_\mu^d \rightarrow V_\nu^d$  we define two operators

$$\sigma_B : \ell^2(V_\nu^d) \rightarrow \ell^2(V_\mu^d), \quad \sigma_B^* : \ell^2(V_\mu^d) \rightarrow \ell^2(V_\nu^d)$$

in the following way

$$\begin{aligned} \sigma_B \varphi(x) &= \varphi(xB); \\ \sigma_B^* \psi(y) &= \begin{cases} \psi(B^{-1}(y)), & \text{if } y \in \operatorname{im} B; \\ 0, & \text{if } y \notin \operatorname{im} B. \end{cases} \end{aligned}$$

In fact, in the previous section we proved the following lemma:

**Lemma 3.6** *Any operator  $W(T)$  can be decomposed as product of the form*

$$W(T) = W(g_1) \lambda_B^* W(g_2) \lambda_C W(g_3),$$

where  $g_1, g_2, g_3$  are elements of symplectic groups and  $B, C$  are appropriate embeddings.

**3.6. Gaussian operators are bounded.** Consider a Gaussian operator  $G(H, Q)$  given by (1.3). Consider projections  $\pi_1 : H \rightarrow V_\mu^d \oplus 0$ ,  $\pi_2 : H \rightarrow 0 \oplus V_\nu^d$ . We represent  $H$  as a finite disjoint union of affine subspaces  $Z_j$  such that  $\pi_1, \pi_2$  are injective on  $Z_j$ . Consider operators

$$G_j f(x) = \sum_{y: (y, x) \in Z_j} \operatorname{Exp}(Q(x, y)) f(y).$$

Actually, for each  $x$  the sum consists of 0 or 1 element. Clearly,  $\|G_j\| = 1$ , and  $G(H, Q) = \sum G_j$ .

**3.7. Products of Gaussian operators.**

**Lemma 3.7** *Let  $X$  be a finite-dimensional space,  $Y$  be a sum of a finite or countable number of copies of  $\mathbb{F}$ . Let  $Q$  be a quadratic form on  $X \times Y$ . Consider the sum*

$$F(y) = \sum_{x \in X} \text{Exp}(Q(x, y)). \quad (3.1)$$

*Then there is a subspace  $Z \subset Y$  of codimension  $\leq \dim X$ , a nonzero constant  $c$ , and a quadratic form  $R$  on  $Z$  such that*

$$F(y) = \begin{cases} c \cdot \text{Exp}(R(y)), & \text{if } y \in Z; \\ 0, & \text{if } y \notin Z. \end{cases}$$

PROOF. This follows from the following observation (see, e.g., [12], Sect. 9.2). Let  $X, U$  be finite-dimensional spaces over  $\mathbb{F}$  of the same dimension. Consider a quadratic form  $P(x)$  on  $X$ . Let

$$f(u) = \sum_{x \in X} \text{Exp}(P(x) + \sum u_j y_j).$$

Then there is a subspace  $K \subset U$  and a quadratic form  $S$  on  $U$  such that

$$f(u) = \begin{cases} c \cdot \text{Exp}(S(u)), & \text{if } u \in K; \\ 0, & \text{if } u \notin K. \end{cases} \quad (3.2)$$

We represent  $Q(x, y)$  as

$$Q(x, y) = Q(x, 0) + \sum_j x_j l_j(y) + Q(0, y),$$

where  $l_j$  are linear forms on  $Y$ , and apply formula (3.2).  $\square$

**Lemma 3.8** *A product of Gaussian operators is a Gaussian operator up to a nonzero scalar factor.*

PROOF. Consider Gaussian operators

$$G(H, Q) : \ell^2(V_\mu^d) \rightarrow \ell^2(V_\nu^d), \quad G(K, R) : \ell^2(V_\nu^d) \rightarrow \ell^2(V_\varkappa^d).$$

We consider the set  $Z$  of all triples  $(u, v, w) \in V_\mu^d \oplus V_\nu^d \oplus V_\varkappa^d$ , such that  $(u, v) \in H$ ,  $(v, w) \in K$ . The kernel of the product is given by

$$N(u, w) = \sum_{v: (u, v, w) \in Z} \text{Exp}(Q(u, v)) \text{Exp}(R(v, w)).$$

We get a sum of the form (3.1). More precisely, consider the natural projection  $Z \rightarrow V_\mu^d \oplus V_\varkappa^d$ . Denote by  $X$  its kernel (it is finite-dimensional), let  $Y$  be a subspace complementary to the  $X$ . We apply Lemma 3.7 and obtain a Gaussian expression for the kernel  $N$ .  $\square$

**Corollary 3.9** *For any  $g \in \text{Sp}(\mathbb{V}_{2\mu})$  operators  $W(g)$  are Gaussian.*

PROOF. Indeed, for generators of  $\text{Sp}(\mathbb{V}_{2\mu})$  the operators  $W(g)$  are Gaussian, see Subs. 2.5. Their products are Gaussian.  $\square$

**Lemma 3.10** *For any perfect Lagrangian linear relation  $T$ , the operator  $W(T)$  is Gaussian.*

PROOF. We refer to Lemma 3.6.  $\square$

### 3.8. End of proof Theorem 1.6.

**Lemma 3.11** *Any Gaussian operator has the form  $W(T)$ .*

PROOF. Consider a Gaussian operator  $G(H; Q) : \ell^2(V_\mu^d) \rightarrow \ell^2(V_\nu^d)$ . Extend the quadratic form  $Q$  to  $V_\mu^d \oplus V_\nu^d$  in an arbitrary way. Represent  $Q$  as

$$Q(x, y) = Q(x, 0) + Q(0, y) + \sum s_{kl} x_k y_l.$$

Let  $C(x)$  be a quadratic form on a space  $V_\pi^d$ . Denote by  $G[[C]]$  the operator in  $\ell^2(V_\pi^d)$  given by

$$G[[C]]\varphi(x) = \text{Exp}(C(x))\varphi(x)$$

Recall (see Subs. 2.5, formula (2.4)), that such operators have a form  $W(g)$  for certain  $g \in \text{Sp}(\mathbb{V}_{2\pi})$ . Consider a Gaussian operator

$$\mathcal{G} := G[[-Q(x, 0)]] G[H; Q] G[[-Q(0, y)]].$$

Clearly, the statements 'the operator  $G(H, Q)$  has a form  $W(\cdot)$ ' and 'the operator  $\mathcal{G}$  has a form  $W(\cdot)$ ' are equivalent. The operator  $\mathcal{G}$  has a form

$$\mathcal{G}\psi(x) = \sum_{y: (x, y) \in H} \text{Exp}\left(\sum s_{kl} x_k y_l\right) \psi(y) =: \sum_{y: (y, x) \in H} \text{Exp}(x S y^t) \psi(y).$$

Let us describe the set  $T$  of all  $(v, w) \in \mathbb{V}_{2\mu} \oplus \mathbb{V}_{2\nu}$  such that

$$a(w)\mathcal{G} = \mathcal{G}a(v).$$

Consider  $(p, q) \in H$ . Then

$$((p, qS); (q, -pS)) \in T. \quad (3.3)$$

Next, let  $(\xi, \eta) \in V_\mu^c \oplus V_\nu^c$  be contained in  $H^\circ$ . Then

$$((0, \xi); (0, \eta)) \in T. \quad (3.4)$$

It is easy to see that elements of forms (3.3) and (3.4) generate a perfect Lagrangian relation.  $\square$

### 3.9. Products of perfect Lagrangian relations. Proof of Theorem 1.4.

**Lemma 3.12** Fix a perfect Lagrangian relation  $T$ . Let  $(\beta, \gamma)$  satisfy

$$a(\gamma)W(T) = W(T)a(\beta).$$

Then  $(\beta, \gamma) \in T$ .

PROOF. Let  $p, q \in \mathbb{V}_{2\infty}$ . Then

$$a(p)a(q)a(-p)a(-q) = \text{Exp}(\{p, q\}).$$

Let  $(\beta, \gamma)$  does not contained in  $T$ . Choose a vector  $(u, v) \in T$  that is not orthogonal to  $(\beta, \gamma)$ . This means that  $\{\beta, u\} \neq \{\gamma, v\}$ . For a field  $\mathbb{F}$  of a prime order this implies

$$\text{Exp}(\{\beta, u\}) \neq \text{Exp}(\{\gamma, v\}). \quad (3.5)$$

For an arbitrary finite field we can multiply  $(u, v)$  by a constant factor, in this way we can achieve (3.5). Next, we have

$$a(\beta)a(\gamma)a(-\beta)a(-\gamma)W(T) = W(T)a(u)a(v)a(-u)a(-v),$$

or

$$\text{Exp}(\{\beta, u\}) \cdot W(T) = \text{Exp}(\{\gamma, v\}) \cdot W(T)$$

Hence  $W(T) = 0$ , and we came to a contradiction.  $\square$

**Lemma 3.13** Let  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$ ,  $S : \mathbb{V}_{2\nu} \rightrightarrows \mathbb{V}_{2\kappa}$  be perfect Lagrangian linear relations. Then the linear relation  $ST$  satisfies the following properties:

a) The linear relation  $ST$  is isotropic and for any  $(v, w) \in ST$  we have

$$a(w)W(S)W(T) = W(S)W(T)a(v). \quad (3.6)$$

b)  $\ker ST$ ,  $\text{indef } ST$  are compact, and  $\text{dom } ST$ ,  $\text{im } ST$  are codiscrete.

c)  $ST$  is contained in a certain perfect Lagrangian relation  $R$  such that  $W(R) = W(S)W(T)$ .

PROOF. a) Let  $(v, w), (v', w') \in ST$ . Choose  $u, u' \in \mathbb{V}_{2\nu}$  such that  $(v, u), (v', u') \in T$  and  $(u, w), (u', w') \in S$ . Since  $T, S$  are isotropic, we have

$$\{u, u'\} = \{v, v'\} = \{w, w'\}.$$

Thus  $ST$  is isotropic.

Next,

$$W(S)W(T)a(v) = W(S)a(u)W(T) = a(w)W(S)W(T).$$

b) The subspace  $\ker ST$  is the set of all  $v$  such that there is  $u$  satisfying  $u \in \ker T$ ,  $(v, u) \in S$ . Thus we take the preimage  $Z$  of  $\ker T$  under the projection  $T \rightarrow V_\mu \oplus 0$ , and  $\ker ST$  is the projection of  $Z$  to  $0 \oplus V_\mu$ . Fibers of the projection  $T \rightarrow V_\mu \oplus 0$  are compact, therefore  $Z$  is compact, and  $\ker ST$  is compact.

Let us verify the statement about  $\text{im } ST$ . The subspace  $\text{im } T \cap \text{dom } S$  is codiscrete in  $\text{dom } S$ . Its image  $H$  under the projection  $\text{dom } S \rightarrow \text{dom } S / \ker S$  also is codiscrete. The relation  $S$  determines a symplectic isomorphism  $\tilde{S} : \text{dom } S / \ker S \rightarrow \text{im } S / \text{indef } S$ . The subspace  $\tilde{S}H$  is codiscrete in  $\text{im } S / \text{indef } S$ . Therefore its lift to  $\text{im } S$  (it is  $\text{im } ST$ ) is codiscrete in  $\text{im } S$  and therefore in  $\mathbb{V}_{2\kappa}$ .

c) Operators  $W(T)$  and  $W(S)$  are Gaussian. Therefore,  $W(S)W(T)$  is Gaussian, and hence it has a form  $W(R)$ .  $\square$

**Lemma 3.14** *Let  $X, Y$  be codiscrete coisotropic subspaces in  $\mathbb{V}_{2\infty}$ . Consider the symplectic space  $X/X^\diamond$ , the image  $K$  of  $X \cap Y \subset X$  in  $X/X^\diamond$  and the image  $L$  of  $X \cap Y^\diamond \subset X$  in  $X/X^\diamond$ . Then  $L = K^\diamond$  in  $X/X^\diamond$ . Moreover,  $L$  is compact, and  $K$  is codiscrete.*

PROOF. We have

$$\begin{aligned} K &= \tilde{K}/X^\diamond, \quad \text{where} \quad \tilde{K} = (X \cap Y) + X^\diamond; \\ L &= \tilde{L}/X^\diamond, \quad \text{where} \quad \tilde{L} = (X \cap Y^\diamond) + X^\diamond. \end{aligned}$$

The space  $X$  is equipped with a skew-symmetric bilinear form, whose kernel is  $X^\diamond$ . It is clear that  $\tilde{K}$  and  $\tilde{L}$  are orthogonal in  $X$ , therefore  $K$  and  $L$  are orthogonal in  $X/X^\diamond$ . Next,

$$\tilde{K}^\diamond = (X^\diamond + Y^\diamond) \cap X.$$

Let  $h \in \tilde{K}^\diamond$ . Let  $\tilde{h}$  be its representative,  $\tilde{h} = a + b$ , where  $a \in X^\diamond$ ,  $b \in Y^\diamond$ . Then  $b$  also is a representative of  $h$ , and hence  $h \in L$ .  $\square$

**Lemma 3.15** *Let  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$  be a perfect Lagrangian linear relation. Let  $R \subset T$  be a relation  $\mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$ . Assume that*

$$(\ker R)^\diamond = \text{dom } R, \quad (\text{dom } R)^\diamond = \ker R,$$

or

$$(\text{im } R)^\diamond = \text{indef } R, \quad (\text{indef } R)^\diamond = \text{im } R.$$

Then  $T = R$ .

PROOF. Let  $R \subsetneq T$ . Considering projection to  $\mathbb{V}_{2\mu} \oplus 0$ , we get

$$\text{dom } R \subsetneq \text{dom } T, \quad \text{or} \quad \text{indef } R \subsetneq \text{indef } T.$$

Therefore,

$$\text{dom } R \subsetneq (\ker T)^\diamond \subset (\ker R)^\diamond,$$

or

$$\text{indef } R \subsetneq \text{indef } T = (\text{im } T)^\diamond \subset (\text{im } R)^\diamond.$$

This contradicts to the conditions of the lemma.  $\square$ ,



PROOF OF THEOREM 1.4. Let  $T : \mathbb{V}_{2\mu} \rightrightarrows \mathbb{V}_{2\nu}$ ,  $S : \mathbb{V}_{2\nu} \rightrightarrows \mathbb{V}_{2\kappa}$  be perfect Lagrangian linear relations. We wish to prove that  $ST$  is perfect Lagrangian.

Without loss of generality we can assume

$$\ker T = 0, \quad \text{indef } S = 0. \quad (3.7)$$

Otherwise, we take the natural projection

$$\text{dom } T \oplus \mathbb{V}_{2\nu} \rightarrow (\text{dom } T / \ker T) \oplus \mathbb{V}_{2\nu}$$

and the image  $\tilde{T}$  of  $T$  under this projection. We get a linear relation  $\tilde{T} : \text{dom } T / \ker T \rightrightarrows \mathbb{V}_{2\nu}$ . Clearly, the statements ' $ST$  is perfect Lagrangian' and ' $S\tilde{T}$  is perfect Lagrangian' are equivalent. In the same way we can assume  $\text{indef } S = 0$ .

Under the condition (3.7),

$$\text{dom } T = \mathbb{V}_{2\mu}, \quad \text{im } S = \mathbb{V}_{2\kappa}.$$

Next,

$$\begin{aligned} \text{im } ST &= S \text{im } T = S(\text{im } T \cap \text{dom } S), \\ \text{indef } ST &= S \text{indef } T = S(\text{indef } T \cap \text{dom } S). \end{aligned}$$

Consider the map  $\text{dom } S \oplus \mathbb{V}_{2\kappa} \rightarrow (\text{dom } S / \ker S) \oplus \mathbb{V}_{2\kappa}$ . Let  $\hat{S}$  be the image of  $S$  under this map. It is a graph of an symplectic bijection  $\sigma : \text{dom } S / \ker S \rightarrow \mathbb{V}_{2\kappa}$ . We have

$$\begin{aligned} \text{im } ST &= \sigma \left[ ((\text{im } T \cap \text{dom } S) + \ker S) / \ker S \right], \\ \text{indef } ST &= \sigma \left[ ((\text{indef } T \cap \text{dom } S) + \ker S) / \ker S \right]. \end{aligned}$$

By Lemma 3.14, the spaces in the square brackets are orthogonal complements one to another. Therefore, the same holds for the left hand sides. By Lemma 3.15,  $ST$  is Lagrangian.  $\square$

**3.10. End of proof Theorem 1.5.** After the establishment of Theorem 1.4 Lemma 3.13.a becomes the statement b) of Theorem 1.5.

To prove the statement c) of Theorem 1.5 we write adjoint operators to the both sides of (1.2),

$$W(P)^* a(-w) = a(-v) W(P)^*,$$

or

$$a(v) W(P)^* = W(P)^* a(w).$$

This is the defining relation for the operator  $W(P^\square)$ .

## References

- [1] Berezin, F. A. *Canonical operator transformation in representation of secondary quantization*. Soviet Physics Dokl. 6, 1961, 212-215.
- [2] Berezin, F. A. *The method of second quantization*. Nauka, Moscow, 1965  
English transl.: Academic Press, New York-London, 1966.
- [3] Friedrichs, K. O. *Mathematical Aspects of the Quantum Theory of Fields*, Interscience Publishers, New York, 1953.
- [4] Gurevich S., Hadani R. *The geometric Weil representation*. Sel. math., New ser. 13 (2008) 465-481.
- [5] Howe, R. *Invariant theory and duality for classical groups over finite fields, with application to singular representation theory*. Preprint, Yale University, 1976.
- [6] Howe, R. (1988), *The Oscillator Semigroup*, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, 48, 61-132
- [7] Morris, S. A. *Pontryagin duality and the structure of locally compact abelian groups*. Cambridge University Press, 1977.
- [8] Nazarov, M. *Oscillator semigroup over a non-Archimedean field*. J. Funct. Anal. 128 (1995), no. 2, 384-438.
- [9] Nazarov, M.L., Neretin, Yu. Olshanski, G. *Semi-groupes engendrés par la représentation de Weil du groupe symplectique de dimension infinie*. C. R. Acad. Sci. Paris Sér. I Math. 309 (1989), no. 7, 443-446.
- [10] Neretin, Yu. A. *On a semigroup of operators in the boson Fock space*. Funct. Anal. Appl. 24 (1990), no. 2, 135-144.
- [11] Neretin, Yu. A. *Categories of symmetries and infinite-dimensional groups*. Oxford University Press, New York, 1996.
- [12] Neretin, Yu. A. *Lectures on Gaussian integral operators and classical groups*. European Mathematical Society (EMS), Zürich, 2011.
- [13] Neretin Yu.A. *The space  $L^2$  on semi-infinite Grassmannian over finite field*. Adv. Math. 250 (2014), 320-350.
- [14] Olshanski, G. I. *On semigroups related to infinite-dimensional groups. Topics in representation theory*, 67-101, Adv. Soviet Math., 2, Amer. Math. Soc., Providence, RI, 1991
- [15] Olshanski, G. I. *Weil representation and norms of Gaussian operators* Functional Analysis and Its Applications, 1994, 28:1, 42-54

- [16] Segal, I. E. *Foundations of the theory of dynamical systems of infinitely many degrees of freedom. I.* Mat.-Fys. Medd. Danske Vid. Selsk. 31, 1959, no. 12,
- [17] Shale, D. *Linear symmetries of free boson fields.* Trans. Amer. Math. Soc. 103, 1962, 149-167.
- [18] Weil, A. *Sur certains groupes d'opérateurs unitaires.* Acta Math. 111, 1964, 143-211.
- [19] Zelenov, E. I. *A  $p$ -adic infinite-dimensional symplectic group.* Izvest. Math. 43 (1994), no. 3, 421-441.

Math.Dept., University of Vienna,  
 Oskar-Morgenstern-Platz 1, 1090 Wien;  
 & Institute for Theoretical and Experimental Physics (Moscow);  
 & Mech.Math.Dept., Moscow State University;  
 & Institute for information transmission problems (Moscow);  
 e-mail: neretin(at) mccme.ru  
 URL: [www.mat.univie.ac.at/~neretin](http://www.mat.univie.ac.at/~neretin)